

FOUR-DIMENSIONAL HAKEN COBORDISM THEORY.

BELL FOOZWELL AND HYAM RUBINSTEIN

ABSTRACT. Cobordism of Haken n -manifolds is defined by a Haken $(n+1)$ -manifold W whose boundary has two components, each of which is a closed Haken n -manifold. In addition, the inclusion map of the fundamental group of each boundary component to $\pi_1(W)$ is injective. In this paper we prove that there are 4-dimensional Haken cobordisms whose boundary consists of any two closed Haken 3-manifolds. In particular, each closed Haken 3-manifold is the π_1 -injective boundary of some Haken 4-manifold.

1. INTRODUCTION

The authors have defined and studied Haken n -manifolds and Haken cobordism theory in previous work [5]. These manifolds enjoy important properties — for example, the universal cover of a closed Haken n -manifold is \mathbf{R}^n (see Foozwell [4]). We would like to know if Haken 4-manifolds are abundant or relatively rare manifolds. We will show that they are abundant in the following sense:

For each pair of closed Haken 3-manifolds M, M' , there is a Haken 4-manifold W with boundary $\partial W = M \cup M'$. In addition, the inclusion maps induce injections $\pi_1(M) \rightarrow \pi_1(W)$ and $\pi_1(M') \rightarrow \pi_1(W)$. The special case when $M' = \emptyset$ is of particular interest.

Our proof of this result will be obtained in a number of steps. The first step is to show that if M is a torus-bundle over a circle, then there is a Haken 4-manifold W with boundary $\partial W = M$. We do this in section 4. We then show a similar result for general surface-bundles in section 5. To show that Haken manifolds satisfy our main result, we use a result of Gabai [6] and Ni [12] in section 6.

It is well-known that all closed 3-manifolds are null cobordant, i.e. bound compact 4-manifolds. Davis, Januszkiewicz and Weinberger [2] following on from work in [1], show that if an aspherical closed n -manifold is null cobordant, then it bounds an aspherical $(n+1)$ -manifold, and furthermore, the inclusion map of the boundary is π_1 -injective. Haken n -manifolds satisfy the stronger property (than asphericity) that they have universal covering by \mathbf{R}^n , as shown in [4]. Moreover for Haken cobordism theory (see [5]), the inclusion maps of the n -manifolds into the $(n+1)$ -dimensional cobordism are π_1 -injective.

2. ACKNOWLEDGEMENTS

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3. HAKEN n -MANIFOLDS

For simplicity, all manifolds will be assumed to be orientable throughout this paper.

Let W be a compact n -manifold and let \underline{w} be a finite collection of $(n-1)$ -dimensional submanifolds in ∂W . We say that \underline{w} is a *boundary-pattern* if whenever A_1, \dots, A_i is a collection of distinct elements of \underline{w} , then $A_1 \cap \dots \cap A_i$ is an $(n-i)$ -dimensional manifold¹. A boundary-pattern is *complete* if $\partial W = \cup\{A : A \in \underline{w}\}$.

The empty boundary-pattern is a special case of a boundary-pattern, and thus a closed manifold is a manifold with boundary-pattern.

Boundary-patterns arise naturally in splitting situations. Suppose that M is a two-sided codimension-one submanifold of W . Let $W|M$ denote the manifold obtained by splitting W open along M . There is a surjective map $q: W|M \rightarrow M$, that reverses the process of splitting W open along M . We call q the *unsplitting map*. If W has a boundary-pattern \underline{w} , then B is an element of the *natural boundary-pattern* of $W|M$ if either

- B is a component of $q^{-1}(A)$ for some $A \in \underline{w}$, or
- B is a component of $q^{-1}(M)$.

Suppose that (W, \underline{w}) is a manifold with boundary-pattern, that $g: D^2 \rightarrow \partial W$ is a piecewise-linear map and that the image of g is contained in at most three elements of \underline{w} . If $g(D^2)$ is contained in a single element of \underline{w} , say A_1 , then $g^{-1}(A_1) = D^2$ and we say that the *preimage of g is a type 1 disk*. If $g(D^2)$ is contained in two distinct elements of \underline{w} , say A_1 and A_2 , then $g^{-1}(A_1) \cup g^{-1}(A_2) = D^2$. If both $g^{-1}(A_1)$ and $g^{-1}(A_2)$ are disks, then we say that the *preimage of g is a type 2 disk*. If $g(D^2)$ is contained in three distinct elements of \underline{w} , say A_1, A_2 and A_3 , then $g^{-1}(A_1) \cup g^{-1}(A_2) \cup g^{-1}(A_3) = D^2$. If each of $g^{-1}(A_1)$, $g^{-1}(A_2)$ and $g^{-1}(A_3)$ are disks, then we say that the *preimage of g is a type 3 disk*. Type j disks are illustrated in figure 1 for $j \in \{1, 2, 3\}$.

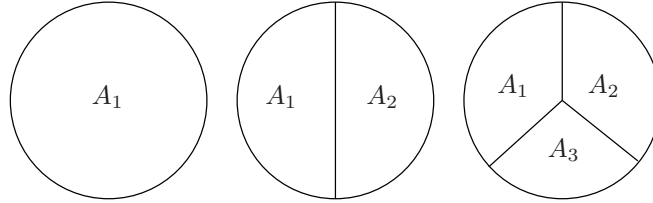


FIGURE 1. Disks of types 1, 2 and 3.

Let $j \in \{1, 2, 3\}$ and let $f: D^2 \rightarrow W$ be a map such that $f(\partial D^2)$ is a loop in ∂W that meets j distinct elements of the boundary-pattern. Suppose

¹The only manifold of negative dimension is the empty set. The empty set is also a manifold in each non-negative dimension.

that for each such map f there is a map $g: D^2 \rightarrow \partial W$ homotopic to f rel ∂ such that the preimage of g is a type j disk. Then we say that \underline{w} is a *useful* boundary-pattern.

In his solution to the word problem, Waldhausen [14] showed that the boundary-patterns that arise in splitting situations for Haken 3-manifolds can always be modified to be useful. (Note that boundary patterns were formally introduced later by Johannson in [7] — they were not explicitly mentioned in [14]).

A map between manifolds with boundary-patterns should relate the boundary-patterns in a reasonable way. We use the following definition. If (W, \underline{w}) and (V, \underline{v}) are manifolds with boundary-patterns, then an *admissible map* is a continuous function $f: W \rightarrow V$ that is transverse to the boundary-patterns and satisfies

$$\underline{w} = \bigsqcup_{A \in \underline{v}} \{B : B \text{ is a component of } f^{-1}(A)\}.$$

We write $f: (W, \underline{w}) \rightarrow (V, \underline{v})$ to indicate that the map f is admissible. Admissible homeomorphisms, embeddings and so on are defined in the obvious way.

If a properly embedded arc can be pushed into the boundary-pattern so that it is contained in no more than two boundary-pattern elements, then we say that the arc is *inessential*. More precisely, let (J, \underline{j}) be a compact 1-dimensional manifold with complete boundary-pattern and let $\sigma: (J, \underline{j}) \rightarrow (W, \underline{w})$ be an admissible map. We say that σ is an *inessential curve* if there is an admissible map $g: (\Delta, \underline{\delta}) \rightarrow (W, \underline{w})$ such that

- (1) $J = \text{Cl}(\partial\Delta \setminus \bigcup\{A : A \in \underline{\delta}\})$
- (2) $\underline{\delta}$ consists of at most two elements,
- (3) $g|_J = \sigma$.

The boundary-pattern $\underline{\delta}$ consists of one element if both endpoints of σ are contained in the same element of \underline{w} , consists of two elements if the endpoints of σ are contained in distinct elements of \underline{w} . If J is a circle, then $\underline{\delta}$ is empty. We say that $\sigma: (J, \underline{j}) \rightarrow (W, \underline{w})$ is an *essential curve* if there is no map $g: (\Delta, \underline{\delta}) \rightarrow (W, \underline{w})$ satisfying the three properties above.

An admissible map $f: (M, \underline{m}) \rightarrow (W, \underline{w})$ is *essential* if each essential curve $\sigma: (J, \underline{j}) \rightarrow (M, \underline{m})$ defines an essential curve $f \circ \sigma: (J, \underline{j}) \rightarrow (W, \underline{w})$. Let M be a submanifold of W . We say that (M, \underline{m}) is an *essential submanifold* of (W, \underline{w}) if the inclusion map is admissible and essential.

Let (W, \underline{w}) be an n -manifold with complete and useful boundary-pattern and let (M, \underline{m}) be a two-sided codimension-one submanifold of W for which the inclusion map is admissible and essential. Then we say that (W, M) is a *good pair*.

A *Haken 1-cell* is an arc with complete and useful boundary-pattern. If $n > 1$, then a *Haken n -cell* is an n -cell with complete and useful boundary-pattern such that each element of the boundary-pattern is a Haken $(n-1)$ -cell.

Let (W_0, \underline{w}_0) be an n -manifold with complete and useful boundary-pattern. A finite sequence of good pairs

$$(W_0, M_0), (W_1, M_1), \dots, (W_k, M_k)$$

is called a *hierarchy* if

- (1) W_{i+1} is obtained by splitting W_i open along M_i ,
- (2) W_{k+1} is a finite disjoint union of Haken n -cells.

A manifold with a hierarchy is called a *Haken n -manifold*.

By definition, each element of the boundary-pattern of a Haken n -manifold is π_1 -injective. By convention, when we say that a manifold is Haken without explicitly referring to a boundary-pattern, the boundary-pattern is simply the disjoint union of the boundary components. For example, suppose that a manifold W has two boundary components, X and Y . If we assert that W is a Haken manifold, then this is meant to imply that X and Y are π_1 -injective in W and that the boundary-pattern of W is $\{X, Y\}$.

Fibre-bundles that have aspherical surfaces as base and fibre provide examples of Haken 4-manifolds. The hierarchy is obtained by lifting essential curves and arcs in the base surface to the 4-manifold. These manifolds will play an important role in this paper.

Let $\underline{w} = \{M_1, \dots, M_j\}$ be a finite collection of closed Haken n -manifolds. If W is a connected Haken $(n+1)$ -manifold with boundary-pattern \underline{w} , then we say that W is a *Haken cobordism*. If the collection \underline{w} consists of just two manifolds, then we may regard a Haken cobordism as an equivalence relation between Haken n -manifolds.

Our interest is in Haken cobordism as a relation between Haken 3-manifolds. In section 6, we will give a condition for two connected Haken 3-manifolds to form the boundary of a Haken cobordism. We will also show that each closed Haken 3-manifold is the boundary of some Haken 4-manifold. As a first step, the following lemma was proved in Foozwell's thesis [3].

Lemma 3.1. *If N is obtained from the Haken 3-manifold M by splitting M open along an incompressible surface F and re-gluing the boundary components, then there is a Haken 4-manifold W with $\partial W = M \sqcup N \sqcup E$, where E is a surface-bundle over the circle with fibre F .*

The idea of the proof is to form $M \times [0, 1]$ and attach a copy of $R(F) \times [0, 1]$ to a regular neighbourhood of parallel copies of $F \times \{1\}$ in $M \times \{1\}$ as indicated in figure 2. We denote by $R(F)$ the regular neighbourhood of F . It is easy to see that the right boundary components are obtained, and with a little more work² one can see that W is indeed a Haken 4-manifold.

After dealing with bundles in the next two sections, we will see how to improve upon lemma 3.1.

4. TORUS-BUNDLES

In this section, we show that each torus-bundle over the circle is the boundary of some Haken 4-manifold.

²The extra work consists of showing that each boundary component is π_1 -injective, and showing that a hierarchy exists.

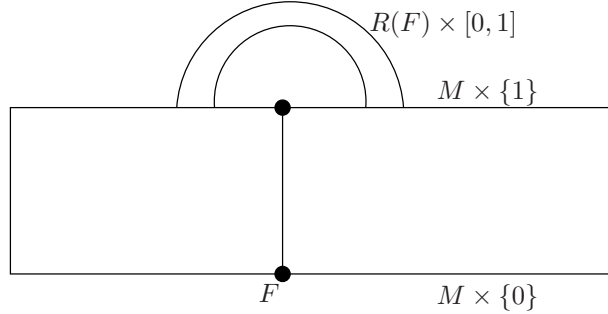


FIGURE 2. Building a Haken 4-manifold with three boundary components.

We first introduce some conventions of notation and orientation that will be used throughout this paper.

If $g: S \rightarrow S$ is a homeomorphism of a surface S , then $S(g)$ is the surface-bundle over the circle with fibre S and monodromy g . More concretely,

$$(1) \quad S(g) = S \times [0, 1] / (x, 0) \sim (g(x), 1).$$

We will use the above notation for fibre-bundles throughout this paper.

The following conventions regarding orientations on manifolds and their boundaries will be used. If S is an orientable surface, then an orientation for S can be specified by an ordered linearly independent pair of vectors (w, x) at a single point $p \in S$. The standard orientation for $S(g)$ is then (w, x, y) where y is a non-zero vector based at $(p, 0)$ tangent to $\{p\} \times [0, 1]$ and directed towards 1. The standard orientation of the 4-manifold $S(g) \times [0, 1]$ is (w, x, y, z) where z is a non-zero vector based at $(p, 0, 0)$ tangent to $\{(p, 0)\} \times [0, 1]$ and directed towards 1. We write the boundary of $S(g) \times [0, 1]$ as

$$(2) \quad \partial(S(g) \times [0, 1]) = S(g^{-1}) \sqcup S(g).$$

Since $S(g^{-1})$ is homeomorphic to $S(g)$, with a reversal of orientation, we use the term $S(g^{-1})$ in expression (2) to represent the manifold $S(g) \times \{0\}$ with the orientation induced by the outward normal convention. The term $S(g)$ in expression (2) represents the manifold $S(g) \times \{1\}$, also with the outward normal convention.

Example 4.1. Let $T^2(\varphi)$ be the torus-bundle over a circle with monodromy φ a single Dehn twist. We represent the torus-bundle $T^2(\varphi)$ by considering

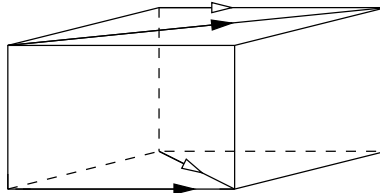


FIGURE 3. Torus bundle with single Dehn twist.

the torus as the square $[0, 1] \times [0, 1]$ in the plane with sides identified in the usual way. The monodromy φ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We

represent $T^2(\varphi)$ visually in figure 3, regarding $T^2(\varphi)$ as the quotient space $(T^2 \times [0, 1]) / (x, 0) \sim (\varphi(x), 1)$.

We consider a special case of lemma 3.1 that we will use subsequently. Let $W_1 = T^2(\varphi) \times [0, 1]$, which is a torus-bundle over an annulus. The boundary of W_1 is $(T^2(\varphi) \times \{0\}) \sqcup (T^2(\varphi) \times \{1\})$. Let us pick out two disjoint parallel torus fibres in $T^2(\varphi) \times \{1\}$. These are: $T_i = T^2 \times \{i/3\} \times \{1\}$ for $i = 1$ or 2 . Let ε be a sufficiently small positive number³, and consider the ε -neighbourhoods of these tori: $T_i(\varepsilon) = T^2 \times [i/3 - \varepsilon, i/3 + \varepsilon] \times \{1\}$. Attach a copy of $T^2 \times [-\varepsilon, \varepsilon] \times [0, 1]$ to $T^2(\varphi) \times \{1\}$ so that $T^2 \times [-\varepsilon, \varepsilon] \times \{0\}$ meets $T_1(\varepsilon)$ and $T^2 \times [-\varepsilon, \varepsilon] \times \{1\}$ meets $T_2(\varepsilon)$. We choose the attachment so that the boundary of the resulting manifold W is

$$T^2(\varphi^{-1}) \sqcup T^2(\psi) \sqcup T^2(\varphi \circ \psi)$$

where $\psi \in \text{SL}(2, \mathbf{Z})$. The manifold W is an orientable Haken 4-manifold with three boundary components. The orientations on the boundary components is based on the orientation convention in expression (2). If we regard ψ as a product of k Dehn twists, then this example shows how to construct a Haken cobordism between torus-bundles with $k + 1$ Dehn twists, k Dehn twists and a single Dehn twist.

Theorem 4.2. *If M is a torus-bundle over a circle, then there is a Haken 4-manifold W with boundary $\partial W = M$.*

We will prove theorem 4.2 via a sequence of lemmas. The first of these is a simple observation that is probably well-known.

Lemma 4.3. *Let F and G be closed orientable incompressible surfaces in a closed orientable 3-manifold M . Suppose that $F \cap G$ is a simple closed curve α . The manifold obtained by splitting M open along F and regluing via a Dehn twist along α is homeomorphic to the manifold obtained by splitting M open along G and regluing via a Dehn twist along α .*

Proof. The result of either operation is simply Dehn surgery on the curve α . \square

Lemma 4.4. *If M_φ is a torus-bundle over a circle with monodromy φ a single Dehn twist, then there is a Haken 4-manifold W with boundary $\partial W = M_\varphi$.*

Proof. Let Σ be a closed orientable surface of genus three. We may regard Σ as the double of the thrice-punctured disk, which is shown in figure 4. Three of the four boundary components of the thrice-punctured disk are labelled in figure 4. Let ϵ_i be a curve parallel to the boundary component labelled i in figure 4. The curve ϵ_4 is parallel to the unlabelled boundary component. Let α , β and γ be the curves shown in figure 4. Up to isotopy, the identity mapping $\text{id}: \Sigma \rightarrow \Sigma$ can be written as a product of three positive Dehn twists and four negative Dehn twists. This observation is a consequence of the lantern relation [8] of the mapping class group. Let f_α be the right-handed Dehn twist about α , and define f_β and f_γ similarly. Let f_i be the right handed Dehn twist about ϵ_i . The lantern relation is

$$f_\gamma f_\beta f_\alpha = f_1 f_2 f_3 f_4.$$

³The number ε is sufficiently small in the sense that $T_1(\varepsilon) \cap T_2(\varepsilon) = \emptyset$.

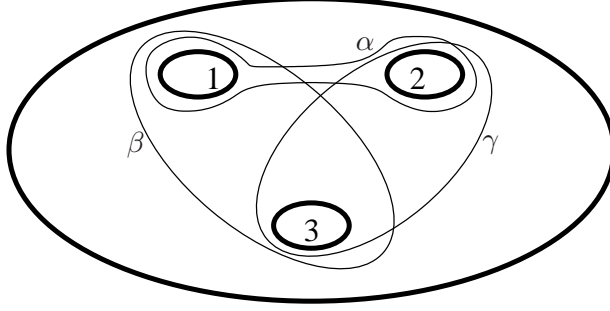


FIGURE 4. Lantern relation.

We may write this relation in a number of ways; each ϵ_i is disjoint from the other curves, so for example, f_i commutes with the other Dehn twists. Thus, up to isotopy, we may write the identity mapping as

$$f_1^{-1} f_\gamma f_2^{-1} f_\beta f_3^{-1} f_\alpha f_4^{-1}.$$

We define the following maps:

$$\begin{aligned} \theta_7 &= f_1^{-1} f_\gamma f_2^{-1} f_\beta f_3^{-1} f_\alpha f_4^{-1} & \theta_3 &= \theta_4 f_\beta^{-1} \\ \theta_6 &= \theta_7 f_4 & \theta_2 &= \theta_3 f_2 \\ \theta_5 &= \theta_6 f_\alpha^{-1} & \theta_1 &= \theta_2 f_\gamma^{-1} \\ \theta_4 &= \theta_5 f_3 & \theta_0 &= \theta_1 f_1 \end{aligned}$$

We now show how to construct a Haken 4-manifold W_7 with three surface-bundle boundary components. Specifically,

$$\partial W_7 = \Sigma(\theta_7^{-1}) \sqcup \Sigma(\theta_6) \sqcup T^2(\varphi).$$

The boundary components are written using representatives from the appropriate orientation-preserving homeomorphism class. The orientations of the boundary components are in accordance with the convention from expression (2).

To see how to build W_7 , first note that, by the lantern relation, θ_7 is isotopic to the identity, so $\Sigma(\theta_7) = \Sigma \times S^1$. Then observe that $\Sigma \times S^1$ is related to $\Sigma(\theta_6)$ by splitting open along a fibre and regluing by a Dehn twist along the curve ϵ_4 in the fibre. There is an incompressible torus T in $\Sigma \times S^1$ that intersects the fibre in the curve ϵ_4 . By lemma 4.3, we can also obtain $\Sigma(\theta_6)$ by splitting $\Sigma \times S^1$ open along T and regluing with a Dehn twist. Then lemma 3.1 tells us how to construct W_7 ; we attach a manifold of the form $T^2 \times [0, 1] \times [0, 1]$ to a boundary-component of $(\Sigma \times S^1) \times [0, 1]$.

Observe that in $\Sigma(\theta_6)$ there is an incompressible torus that intersects the fibre in the curve α . By attaching a manifold of the form $T^2 \times [0, 1] \times [0, 1]$ to a boundary-component of $\Sigma(\theta_6) \times [0, 1]$ we obtain a Haken 4-manifold W_6 with boundary

$$\partial W_6 = \Sigma(\theta_6^{-1}) \sqcup \Sigma(\theta_5) \sqcup T^2(\varphi^{-1}).$$

Similarly, there is an incompressible torus in $\Sigma(\theta_5)$ that intersects the fibre in the curve ϵ_3 . We then construct a 4-manifold W_5 with boundary

$$\partial W_5 = \Sigma(\theta_5^{-1}) \sqcup \Sigma(\theta_4) \sqcup T^2(\varphi).$$

We continue creating Haken cobordisms with three boundary components. However, we no longer need to find incompressible tori that intersect the lantern curves. Instead, all the boundary components will be Σ -bundles over the circle. Lemma 3.1 produces Haken 4-manifolds W_4 , W_3 , W_2 , and W_1 with boundaries as follows:

$$\begin{aligned}\partial W_4 &= \Sigma(\theta_4^{-1}) \sqcup \Sigma(\theta_3) \sqcup \Sigma(f_\beta^{-1}) & \partial W_2 &= \Sigma(\theta_2^{-1}) \sqcup \Sigma(\theta_1) \sqcup \Sigma(f_\gamma^{-1}) \\ \partial W_3 &= \Sigma(\theta_3^{-1}) \sqcup \Sigma(\theta_2) \sqcup \Sigma(f_2) & \partial W_1 &= \Sigma(\theta_1^{-1}) \sqcup \Sigma(\theta_0) \sqcup \Sigma(f_1)\end{aligned}$$

Note that θ_0 is the identity mapping so $\Sigma(\theta_0) = \Sigma \times S^1$.

So we have seven orientable Haken 4-manifolds each with three boundary components. We can glue these seven manifolds together to form a connected manifold W' with boundary

$$\begin{aligned}\partial W' &= \Sigma(\theta_7^{-1}) \sqcup T^2(\varphi) \sqcup T^2(\varphi^{-1}) \sqcup T^2(\varphi) \sqcup \Sigma(f_\beta^{-1}) \\ &\sqcup \Sigma(f_2) \sqcup \Sigma(f_\gamma^{-1}) \sqcup \Sigma(f_1) \sqcup \Sigma(\theta_0).\end{aligned}$$

The idea is illustrated in figure 5, which schematically shows the manifolds

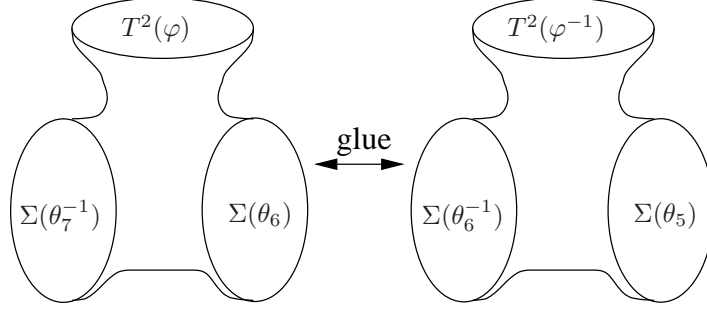


FIGURE 5. Identifying the $\Sigma(\theta_6)$ boundary-components of W_7 and W_6 to produce a connected 4-manifold.

W_7 and W_6 being joined together.

We glue eight of these boundary components in pairs, leaving just one boundary component $T^2(\varphi)$. That is, we glue $T^2(\varphi^{-1}) \subset W_6$ to $T^2(\varphi) \subset W_5$, glue $\Sigma(f_\beta^{-1})$ to $\Sigma(f_2)$ and glue $\Sigma(f_\gamma^{-1})$ to $\Sigma(f_1)$. We also glue $\Sigma(\theta_7)$ to $\Sigma(\theta_0)$. This can all be done so that the result is orientable. Hence there is an orientable Haken 4-manifold W with boundary $\partial W = T^2(\varphi)$. \square

Lemma 4.5. *If M_ψ is a torus-bundle over a circle with monodromy ψ a product of a finite number of Dehn twists, then there is a Haken 4-manifold W with boundary $\partial W = M_\psi$.*

Proof. The construction is similar to that of example 4.1 and is by induction on the number of Dehn twists, say k . Write the monodromy as $\psi = \tau \circ \sigma$ where τ is a product of $k - 1$ Dehn twists and σ is a Dehn twist. We modify the torus-bundle $M_\psi \times [0, 1]$ by attaching a copy of $T^2 \times [-\varepsilon, \varepsilon] \times [0, 1]$ to ε -neighbourhoods of disjoint torus fibres in $M_\psi \times \{0\}$ as in example 4.1, except we choose the gluing so that the boundary components are M_ψ , M_τ and M_σ .

Since σ is a single Dehn twist, we can glue on the compact 4-manifold found in lemma 4.4 to fill in the boundary component M_ψ . We obtain a manifold W with two boundary components M_ψ, M_τ . It is easy to see that W is a Haken 4-manifold. The proof now follows by induction since τ is a product of $k - 1$ Dehn twists. So we can find a Haken 4-manifold with boundary M_τ and glue this onto W to build the required Haken 4-manifold with boundary M_ψ .

Note that the case $k = 1$ follows from lemma 4.4. \square

Putting the results of the lemmas in this section together constitutes a proof of theorem 4.2.

5. HIGHER GENUS SURFACE-BUNDLES

We will use lemma 3.1 in our proof of the main theorem of this section.

Theorem 5.1. *If M is a closed surface-bundle over a circle, then there is a Haken 4-manifold W with $\partial W = M$.*

Proof. As before, the proof is by induction on the number of Dehn twists needed to represent the monodromy. To prove theorem 5.1, we must construct a Haken 4-manifold whose boundary is a surface-bundle with given monodromy.

To start the induction, let F be a closed orientable surface of genus at least two, and let M_φ be the surface bundle $F(\varphi)$, where φ is a Dehn twist along an essential curve α in F . It is clear that we can construct M_φ from the product $F \times S^1$ by cutting $F \times S^1$ open along the fibre $F \times \{p\}$ and regluing with a Dehn twist. By lemma 4.3, we can construct M_φ by splitting $F \times S^1$ along an incompressible torus containing α and regluing with a Dehn twist.

The manifold M_φ is related to the product manifold $F \times S^1$ by a change in homeomorphism along an incompressible torus. Hence, there is a Haken 4-manifold W_1 with boundary $\partial W_1 = M_\varphi \sqcup (F \times S^1) \sqcup E$ where E is the total space of a torus-bundle over a circle. In section 4 we showed that E is the boundary of a Haken 4-manifold, W_2 . The product $F \times S^1$ is also the boundary of a Haken 4-manifold. For example, take a Haken 3-manifold N with boundary $\partial N = F$. Then $N \times S^1$ will suffice. We attach W_2 and $N \times S^1$ to the appropriate boundary components of W_1 to obtain a Haken 4-manifold with boundary M_φ .

To prove the general case, we proceed exactly as in lemma 4.5. Assume that M_φ is a surface bundle over a circle whose monodromy φ is a product of k Dehn twists. Write $\varphi = \tau \circ \psi$ where ψ is a single Dehn twist and τ is a product of $k - 1$ Dehn twists. Using lemma 3.1 and the case above of a surface bundle with monodromy consisting of a single Dehn twist, we can construct a Haken 4-manifold with boundary consisting of the disjoint union of $M_\varphi, M_\tau, M_\psi$ and then glue on a Haken 4-manifold with boundary M_ψ , since ψ is a single Dehn twist. By induction on the number k of Dehn twists, there is another Haken 4-manifold with boundary M_τ since τ is a product of $k - 1$ Dehn twists. Gluing this on completes the proof of the theorem. \square

6. OTHER HAKEN MANIFOLDS

We first prove an extension of lemma 3.1, which gives a sufficient condition for two Haken 3-manifolds to be Haken cobordant.

Theorem 6.1. *If N is obtained from the closed connected Haken 3-manifold M by splitting M open along an incompressible surface F and re-gluing the boundary components, then there is a Haken 4-manifold W with $\partial W = M \sqcup N$, and boundary-pattern $\underline{w} = \{M, N\}$.*

Proof. We use the construction in the proof of lemma 3.1 to obtain a Haken 4-manifold X with boundary $\partial X = M \sqcup N \sqcup E$ and boundary-pattern $\underline{x} = \{M, N, E\}$, where E is a surface-bundle over a circle with fibre F . By theorems 4.2 and 5.1, there is another Haken 4-manifold Y with boundary $\partial Y = E$ and boundary-pattern $\underline{y} = \{E\}$. We form a quotient space of $X \sqcup Y$ by gluing the E boundary components together via a homeomorphism to obtain the required Haken 4-manifold W . \square

Gabai [6] announced the following result in 1983 with an outline of the proof, and recently Ni [12] has provided the details of the proof.

Theorem 6.2. *Let M_1 be a closed Haken 3-manifold. There is a sequence*

$$M_1, M_2, M_3, \dots, M_n$$

such that M_{i+1} is obtained from M_i by splitting M_i open along an incompressible surface and re-gluing the boundary components, and M_n is a product $\Sigma \times S^1$, where Σ is a closed surface.

Using theorem 6.2, we can show that every pair of closed Haken 3-manifolds is the boundary of some Haken 4-manifold.

Theorem 6.3. *Let M, M' be a pair of closed Haken 3-manifolds. Then there is a Haken 4-manifold W with $\partial W = M \sqcup M'$ and boundary-pattern $\underline{w} = \{M, M'\}$.*

Proof. Write $M = M_1$ and using the notation of theorem 6.2 we have a sequence

$$M_1, M_2, M_3, \dots, M_n$$

such that M_{i+1} is obtained from M_i by splitting M_i open along an incompressible surface and re-gluing the boundary components, and $M_n = \Sigma \times S^1$, for some closed orientable aspherical surface Σ . Using induction on the number of terms in the sequence, we use theorem 6.1 to obtain a Haken 4-manifold X with boundary $\partial X = M_1 \sqcup M_n$. Similarly (with obvious notation) there is a Haken 4-manifold Y with boundary $\partial Y = M'_1 \sqcup M'_p$, where $M'_p = \Sigma' \times S^1$ and Σ' is a closed orientable aspherical surface. If Σ' is homeomorphic to Σ we can glue X to Y along the product boundary components to obtain the required Haken cobordism. Otherwise, take a Haken 3-manifold N with boundary $\partial N = \Sigma \sqcup \Sigma'$. Then $N \times S^1$ is a Haken 4-manifold with boundary $(\Sigma \times S^1) \sqcup (\Sigma' \times S^1)$. We can then glue X and Y to the appropriate boundary components of $N \times S^1$ to obtain the required Haken cobordism. \square

Corollary 6.4. *If M is a closed Haken 3-manifold, then there is a Haken 4-manifold W with $\partial W = M$ and boundary-pattern $\underline{w} = \{M\}$.*

7. HYPERBOLIC CASE

In Long and Reid [9], it is shown that if a closed hyperbolic 3-manifold M is the totally geodesic boundary of a compact hyperbolic 4-manifold W , then $\eta(M)$ takes an integer value. In [9] M is said to *geometrically bound* W . On the other hand, Meyerhoff and Neumann [11], show that $\eta(N_\alpha)$ takes a dense set of values in \mathbf{R} for the set $\{N_\alpha\}$ of Dehn surgeries on a hyperbolic knot in S^3 . So this implies that ‘generically’ hyperbolic 3-manifolds do not geometrically bound hyperbolic 4-manifolds.

The existence of π_1 -injective 2-tori in the Haken 4-manifolds constructed in Corollary 6.4 gives an obvious obstruction to these 4-manifolds admitting hyperbolic or even strictly negatively curved metrics.

In [10] Long and Reid give examples of n -dimensional hyperbolic manifolds which geometrically bound hyperbolic $(n+1)$ -dimensional hyperbolic manifolds, for all n .

8. SOME QUESTIONS

The Haken 4-manifolds that we have constructed in this paper fall into a special class. In a sense, they are analogues of the graph manifolds of Waldhausen. Other examples of Haken 4-manifolds exist. For example, the hyperbolic 4-manifolds of Ratcliffe and Tschantz [13] are all finitely covered by Haken 4-manifolds (see [5] for a proof of this). In [5], examples of Haken 4-manifolds which admit metrics of strictly negative curvature but which do not admit hyperbolic metrics are given.

Question 8.1. If M is a closed Haken 3-manifold, does there exist a Haken 4-manifold W with $\partial W = M$ and which contains only non-separating sub-manifolds in its hierarchy? (Note that then the complement of the hierarchy is a single 4-cell.)

Question 8.2. Which closed hyperbolic Haken 3-manifolds M geometrically bound hyperbolic Haken 4-manifolds? Are there other obstructions than that in [9] that the eta invariant of M must be an integer? What about the situation if the Haken 4-manifold admits a metric of strictly negative or non-positive curvature? Is it still true that the eta invariant of M must be an integer in this case?

Question 8.3. For $n > 3$, what are the standard cobordism classes for Haken n -manifolds? We say that Haken n -manifolds N and N' belong to the same standard cobordism class if there is an $(n+1)$ -manifold W for which $\partial W = N \sqcup N'$. In private communication, Allan Edmonds has constructed a Haken 4-manifold with odd Euler characteristic, so we know, for example, that Haken 4-manifolds need not be null cobordant.

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TRINITY COLLEGE, THE UNIVERSITY OF MELBOURNE

E-mail address: bfoozwel@trinity.unimelb.edu.au

URL: <https://sites.google.com/site/bellfoozwell/>

DEPARTMENT OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF MELBOURNE

E-mail address: rubin@ms.unimelb.edu.au

URL: <http://www.ms.unimelb.edu.au/~rubin/>